First Semestral Exam 2019
B. Math. (Hons.) IInd year
Algebra III
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Answer Any FIVE Questions.

Q 1.
(a) Let R be a possibly noncommutative ring with the property that $xy = 0$ implies $x = 0$ or $y = 0$. Let m, n be coprime positive integers and let $a, b \in R$ satisfy $a^m = b^m, a^n = b^n$. Prove $a = b$.
(b) Prove that the polynomial $X^{100} - 123123X^{28} + 110$ cannot take the values ± 33 over integers.

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Q 2.
Determine all integer solutions of $x^2 + 2 = y^3$; you may assume that $\mathbb{Z}[\sqrt{-2}]$ is a UFD.

OR

Consider the integral domain $R = \mathbb{Z}[\sqrt{14}]$. Explain why the equality $2 \cdot 7 = \sqrt{14} \cdot \sqrt{14}$ does not contradict the fact that R is a unique factorization domain.

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Q 3. Let R be a ring with unity and let I be a left ideal. Prove that $S = \{r \in R : Ir \subseteq I\}$ is a subring containing I as a 2-sided ideal. Further, show that the ring $End_R(R/I)$ of left R-module endomorphisms of R/I is isomorphic to the opposite of the ring S/I.

OR

If R is a (not necessarily commutative) ring with unity, then show that a left ideal I is a direct summand of R if and only if $I = Re$ with e idempotent.

Q 4. Find abelian groups A, B, C such that there exists a surjection $\theta : B \to C$ such that $Hom(A, B) \to Hom(A, C); \quad \alpha \mapsto \theta \circ \alpha$ is not surjective.

OR

Let R be a commutative ring with unity and let $\mathbb{M}$ be a maximal ideal. For any positive integer n, prove that the ring $R/\mathbb{M}^n$ has a unique maximal ideal and that every non-unit is nilpotent.

Q 5. Let R be a commutative ring with unity and p a prime such that $a^p = a$ for all $a \in R$. Prove that R is isomorphic to a subring of the direct product of copies of $\mathbb{Z}/p\mathbb{Z}$.
Hint. For any prime ideal P, prove that $R/P \cong \mathbb{Z}/p\mathbb{Z}$ and that $Nil(R) = 0$.

OR

If I, J are ideals satisfying $I + J = R$ for a commutative ring R with unity, prove that $IJ = I \cap J$ and that the map

$$R/IJ \to R/I \times R/J; \quad r + IJ \mapsto (r + I, r + J)$$

is an isomorphism of rings.
Deduce an isomorphism of rings $\mathbb{C}[X]/(X^3 - X) \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C}$.

Q 6. If a group G is generated by symbols $x_1, \cdots, x_n$ and r defining relations $f_i = x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_n^{a_{in}} = 1$ for $1 \leq i \leq r$, its abelianization $G/[G, G]$ is the quotient of the free $\mathbb{Z}$-module with basis $\{x_1, \cdots, x_n\}$ modulo the relation matrix $(a_{ij})_{i \leq r, j \leq n}$. Assuming that the group $SL(2, \mathbb{Z})$ is generated by x, y and defining relations $x^4 = 1 = y^6, x^2 = y^3$, determine the structure of the abelian group $SL(2, \mathbb{Z})/[SL(2, \mathbb{Z}), SL(2, \mathbb{Z})]$.

OR

Find the invariant factors in $\mathbb{Q}[X]$ corresponding to the 6×6 diagonal matrix $diag(1, 1, 1, 2, 2, 3)$.